

# Math 221-Midterm 1 Review Guide

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1. Functions: What is a function, domain/range; composition of functions
2. Limits

- The definitions of (two-sided) limits, left limits and right limits.
- $\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = L.$
- $\lim_{x \rightarrow a} f(x)$  does not depend on how  $f$  is defined at  $x = a$ .
- To be able to recognize/create examples where the limit does not exist.

**Example 1**  $\lim_{x \rightarrow 0} \frac{x}{|x|}, \lim_{x \rightarrow 0} \sin(\frac{1}{x}), \lim_{x \rightarrow 0} \frac{1}{x^2}, \lim_{x \rightarrow 0} \frac{1}{x}$  do not exist.

- Computation of limits
  - Compute limits by using limit properties as in section 6, Chapter III.

**Example 2** Compute  $\lim_{x \rightarrow 1} \frac{(2x^2 + 2)^3}{\sqrt{x} + 1}$

- Compute limits by using algebraic manipulations or by using Sandwich Theorem when the limit properties do not apply.

**Example 3** Compute  $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x - 1}, \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3}, \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

**Example 4** Compute  $\lim_{x \rightarrow 0} x^2 \sin(\frac{1}{x^3})$  and  $\lim_{x \rightarrow \infty} \frac{\cos x}{x}$  by using the Sandwich theorem.

- To know that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and to be able to use it to compute limits like  $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$ ,  $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$ ,  $\lim_{x \rightarrow 0} \frac{x^2}{\sin x^2}$ ,  $\lim_{x \rightarrow 0} \frac{\tan x}{x}$ ,  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$ .

### 3. Continuity

- $f(x)$  is continuous at  $a \iff \lim_{x \rightarrow a} f(x)$  exists and equals  $f(a) \iff \lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = f(a)$ .
- $f(x)$  is a continuous function if it is continuous at all points in its domain.
- Polynomials, rational functions,  $\sin x$  and  $\cos x$  are continuous functions.
- If  $f(x), g(x)$  are continuous functions, then so are  $f \pm g, fg, \frac{f}{g}, f(g(x))$ .
- To be able to recognize/create examples of points of discontinuity.

**Example 5**  $\frac{x}{|x|}$  has a jump discontinuity at  $x = 0$ ;  $x \sin(\frac{1}{x})$  has a removable discontinuity at  $x = 0$ .

**Example 6**  $\sin(\frac{1}{x})$  and  $\frac{1}{x^2}$  are discontinuous at  $x = 0$  (not removable).

- To be able to determine when a piecewisely defined function is continuous.

**Example 7** #26, 27, 28 on page 51 of the textbook; also # 29 on page 54 and #30 on page 55.

### 4. Limits involving infinity and Asymptotes

- To understand the concepts of vertical, horizontal and slanted asymptotes.
- To find the horizontal asymptotes of a function  $f(x)$ , we look at  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$ .
- To find the vertical asymptotes of a function  $f(x)$ , we find places  $x = a$  where  $\lim_{x \rightarrow a+} f(x) = \infty$  (or  $-\infty$ ) or  $\lim_{x \rightarrow a-} f(x) = \infty$  (or  $-\infty$ ).
- If  $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = m$  and  $\lim_{x \rightarrow \infty} [f(x) - mx] = n$ , where  $m$  and  $n$  are finite numbers, then  $f(x)$  is asymptotic to  $y = mx + n$  as  $x \rightarrow \infty$ .

- Similarly, if  $\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = m$  and  $\lim_{x \rightarrow -\infty} [f(x) - mx] = n$ , then  $f(x)$  is asymptotic to  $y = mx + n$  as  $x \rightarrow -\infty$ .

## 5. Derivatives and rates of change

- **To understand the definition of derivative as a limit**

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

**Note:** there will be a couple problems on the exam asking you to compute derivatives using the definition as a limit.

**Example 8** Compute the derivative of  $f(x) = 2x + \frac{1}{x}$  at  $x = 1$  by using the definition of the derivative as a limit and find the equation of the tangent line at  $x = 1$ .

- To understand the derivative  $f'(a)$  as the slope of the tangent to the graph of  $f(x)$  at  $x = a$  and the (instantaneous) rate of change of  $f(x)$  at  $x = a$ . To understand the instantaneous rate of change as the limiting value of average rate of change.
- To understand differentiability and how a function can fail to be differentiable.

**Example 9**  $|x|$ ,  $x^{1/3}$ ,  $x^{2/3}$  are continuous but not differentiable at  $x = 0$ .

**Example 10** #11 on page 62 of the textbook.

- If  $f(x)$  is differentiable at  $x = a$ , then  $f$  is continuous at  $x = a$ .
- **To be able to determine when a piecewisely defined function is differentiable.**

**Example 11** #12, 13, 14 on page 63 of the textbook.

- Differentiation rules: power rule, sum rule, constant multiple rule, product rule, quotient rule, chain rule.
- To know that  $\frac{d}{dx} \sin x = \cos x$  and  $\frac{d}{dx} \cos x = -\sin x$ , and to be able to use them to compute derivatives of other trigonometric functions.
- To understand the concept of higher order derivatives.